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SOME ADDITIONAL COMMENTS ON OPTIMAL STRUCTURAL DESIGN UNDER MULTIPLE EIGENVALUE CONSTRAINTS

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Abstract—Recent results [1] concerning necessary and sufficient conditions for local optimality in the case of a dual eigenvalue are extended to eigenvalues of multiplicity $n > 2$.

Let us introduce this Note by recalling [1] that all displacement modes U(x) are normalized in the sense of

$$\int_\tau W_2(\mathrm{U})\,\mathrm{dx} = 1, \tag{1}$$

in which W_2 is a quadratic positive definite form and τ is the domain of integration. We now assume that a (presumably optimal) design $\overline{H}(x)$ is associated with an n-fold eigenvalue $\overline{\lambda}$ and n eigenmodes $\overline{\mathrm{U}}_i(\mathbf{x})$, orthonormalized for convenience in the sense of

$$\int_\tau W_{11}(\overline{\mathrm{U}}_i, \overline{\mathrm{U}}_j)\,\mathrm{dx} = \delta_{ij} \tag{2}$$

and satisfying

$$\int_\tau Q_2(\overline{\mathrm{U}}_i; \overline{H})\,\mathrm{dx} = \overline{\lambda}, \quad i = 1, 2, \ldots n, \tag{3}$$

in which the positive definite quadratic form Q_2 usually represents the specific strain energy density. Let us also assume that there exists a set of constants γ_{ij} such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \frac{\partial Q_{11}}{\partial H} (\overline{\mathrm{U}}_i, \overline{\mathrm{U}}_j; \overline{H}) = k^2, \quad \mathbf{x} \in \tau. \tag{4}$$

We finally postulate that the matrix $[\gamma_{ij}]$ is positive definite.

It has been shown in [1] that eqn (4) is a necessary condition for optimality and represents a generalization of previously obtained optimality conditions (see, e.g. [2–4] covering special cases or restrictive assumptions). It has also been proved in [1] that for $n = 2$ the positive definiteness of $[\gamma_{ij}]$ is necessary and sufficient for local optimality with respect to all design changes $\dot{H}(\mathbf{x})$ which satisfy the constant-volume constraint

$$\int_\tau \dot{H}\,\mathrm{dx} = 0 \tag{5}$$

and which are associated with a separation of the eigenvalues, that is, for which $\dot{\lambda}_1 \neq \dot{\lambda}_2$. The proof is based on the fact that for all such changes

$$\dot{\lambda}_1 \dot{\lambda}_2 < 0, \tag{6}$$

the smallest eigenvalue is therefore necessarily less in the neighborhood of the "optimal" point. An extension of this type of argument to the general case of $n \geqq 2$ is the objective of this Note.

The eigenfunctions $\overline{\mathbf{U}}_i$ are not defined uniquely through eqn (2), and it is possible [1] to select them in such a way as to reduce $[\gamma_{ij}]$ to a diagonal matrix. In view of the assumed positive definiteness of the latter, eqn (4) then becomes

$$\sum_{i=1}^{n} c_i^2 \frac{\partial Q_2}{\partial H}(\overline{\mathbf{U}}_i; \overline{H}) = k^2, \quad \mathbf{x} \in \tau. \tag{7}$$

Let us now assume that there exists another design $h(x)$ which is associated with the same fundamental eigenvalue $\overline{\lambda}$ and with the eigenmode $\mathbf{u}(\mathbf{x})$, that is, for which

$$\int_\tau Q_2(\mathbf{U}; h)\,\mathrm{d}\mathbf{x} \geqq \int_\tau Q_2(\mathbf{u}; h)\,\mathrm{d}x = \overline{\lambda} \quad \forall\, \mathbf{U}(\mathbf{x}). \tag{8}$$

Specifically this implies

$$\int_\tau Q_2(\overline{\mathbf{U}}_i; h)\,\mathrm{d}\mathbf{x} \geqq \overline{\lambda}, \quad i = 1, 2, \ldots n, \tag{9}$$

and therefore

$$\sum_{i=1}^{n} c_i^2 \int_\tau Q_2(\overline{\mathbf{U}}_i; h)\,\mathrm{d}\mathbf{x} \geqq \left(\sum c_i^2\right) \overline{\lambda}. \tag{10}$$

Certain types of structures, notably sandwich structures, satisfy the condition of concavity

$$Q_2(\mathbf{U}; h) \leqq Q_2(\mathbf{U}; \overline{H}) + (h - \overline{H}) \frac{\partial Q_2}{\partial H}(\mathbf{U}; \overline{H}) \quad \forall\, \mathbf{U}. \tag{11}$$

In that case substitution of eqn (11) in the inequality (10) leads to

$$\sum_{i=1}^{n} c_i^2 \int_\tau Q_2(\overline{\mathbf{U}}_i; \overline{H})\,\mathrm{d}\mathbf{x} + \int_\tau (h - \overline{H}) \sum_{i=1}^{n} c_i^2 \frac{\partial Q_2}{\partial H}(\overline{U}_i; \overline{H})\,\mathrm{d}\mathbf{x} \geqq \sum_{i=1}^{n} c_i^2 \overline{\lambda}. \tag{12}$$

Since, by eqn (3), the first sum on the left hand side of (12) equals the right hand side, it follows from eqn (7) that

$$k^2 \int_\tau (h - \overline{H})\,\mathrm{d}\mathbf{x} \geqq 0. \tag{13}$$

In other words, the design $h(\mathbf{x})$ is associated with a volume which is at least as large as that of the design $\overline{H}(\mathbf{x})$, and global optimality is therefore established.

A more restricted sufficiency condition for optimality can be proved in relation to structures which do not necessarily satisfy the concavity condition (11). For this purpose we recall [1] that the changes $\dot{\lambda}_i$, $i = 1, 2, \ldots n$, in the multiple eigenvalue $\overline{\lambda}$ which are associated with a design change $\dot{H}(\mathbf{x})$ are the principal values of the matrix

$$[\dot{\Lambda}_{ij}(\dot{H})] = \left[\int_\tau \frac{\partial Q_{11}}{\partial H}(\overline{\mathbf{U}}_i, \overline{\mathbf{U}}_j; \overline{H})\dot{H}\,\mathrm{d}\mathbf{x}\right] \equiv \left[\int_\tau \Omega_{ij}\,\dot{H}\,\mathrm{d}\mathbf{x}\right]. \tag{14}$$

Let us now multiply Eqn (4) by $\dot{H}$ and integrate over the domain τ. If $\dot{H}$ satisfies the constant-volume restriction eqn (5), then, by eqn (14),

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \dot{\Lambda}_{ij} = 0 \quad \forall \dot{H}, \tag{15}$$

or, after diagonalization of $[\gamma_{ij}]$ and assuming once again that $[\gamma_{ij}]$ is positive definite,

$$c_1^2 \dot{\Lambda}_{11} + c_2^2 \dot{\Lambda}_{22} + \cdots + c_n^2 \dot{\Lambda}_{nn} = 0. \tag{16}$$

It follows that the matrix $[\dot{\Lambda}_{ij}]$ cannot be definite, irrespective of the choice $\dot{H}$. That is, either all its principal values vanish ($H = H_c$, see [1]), or else, if two or more principal values are distinct then the smallest must be negative. In other words, even for $n > 2$ a local optimum has been established within the subspace of design changes for which at least two eigenvalues separate. This result, though not its derivation, is analogous to the result obtained by Prager and Shield [5] in their seminal paper covering optimal design of multi-purpose structures.

The question of the necessity of the positive definiteness of $[\gamma_{ij}]$ is somewhat more complex. We recall from [1] that $\dot{H}_c(x)$ represents the subspace of design changes for which the degree of multiplicity of the eigenvalue is retained, that is, for which $\dot{\lambda}_1 = \dot{\lambda}_2 = \cdots \dot{\lambda}_n \equiv \dot{\lambda}$. In that case $\dot{\Lambda}_{ij} = \dot{\lambda}\, \delta_{ij}$ and, by Eqn (14), $\dot{H}_c$ satisfies

$$\int_\tau \langle \Omega_{11} - \Omega_{22} \rangle \dot{H}_c \, d\mathbf{x} = \cdots = 0$$
$$\int_\tau \langle \Omega_{12} \rangle \dot{H}_c \, d\mathbf{x} = \cdots = 0. \tag{17}$$

Note that in eqn (17) the replacement of Ω_{ij} by

$$\langle \Omega_{ij}(\mathbf{x}) \rangle = \Omega_{ij}(\mathbf{x}) - (\Omega_{ij})_{\text{average}} \tag{18}$$

is justified in view of eqn (5). Equations (17) are equivalent to

$$\int_\tau \langle \Omega_{ij}^d \rangle \dot{H}_c \, d\mathbf{x} = 0, \quad i, j = 1, 2 \cdots n, \tag{19}$$

in which $\langle \Omega_{ij}^d \rangle$ are the deviatoric components of $\langle \Omega_{ij} \rangle$, that is, with the adoption of the summation convention,

$$\langle \Omega_{ij}^d \rangle = \langle \Omega_{ij} \rangle - \frac{1}{n} \langle \Omega_{kk} \rangle \delta_{ij}. \tag{20}$$

Then the most general expression for a design change satisfying eqn (5) is

$$\begin{aligned} \dot{H}(\mathbf{x}) &= \dot{H}_c(\mathbf{x}) + \dot{\alpha}_{kl} \langle \Omega_{kl}^d(\mathbf{x}) \rangle \\ &= \dot{H}_c(\mathbf{x}) + \dot{\alpha}_{kl}^d \langle \Omega_{kl}(\mathbf{x}) \rangle, \end{aligned} \tag{21}$$

in which $\dot{\alpha}_{kl}^d$, the deviatoric part of $\dot{\alpha}_{kl}$, satisfies the obvious restriction

$$\dot{\alpha}_{kk}^d = 0. \tag{22}$$

Let us now insert eqn (21) into eqn (14). Optimality requires that the first part involving H_c vanish; this condition leads [1] to the establishment of eqn (4). What is left is

$$\dot{\Lambda}_{ij} = M_{ijkl} \dot{\alpha}_{kl}^d, \tag{23}$$

with

$$M_{ijkl} = M_{jikl} = M_{klij} = \int_T \langle \Omega_{ij} \rangle \langle \Omega_{kl} \rangle \, dx. \tag{23a}$$

There are $\frac{1}{2}\, n(n+1)$ independent components of $\dot{\Lambda}_{ij}$ and, in view of eqn (22), the number of independent components of $\dot{\alpha}^d_{kl}$ is one less. Barring pathological behavior we may therefore select a nontrivial set of $\dot{\alpha}^d_{kl}$ in such a way that all nondiagonal and all but two diagonal terms in $\dot{\Lambda}_{ij}$ vanish. Let the latter two be $\dot{\Lambda}_{11} = \dot{\lambda}_1$ and $\dot{\Lambda}_{22} = \dot{\lambda}_2$, and let the principal values of $[\gamma_{ij}]$ be γ_i. Instead of eqn (16) we then have

$$\gamma_1 \dot{\lambda}_1 + \gamma_2 \dot{\lambda}_2 = 0, \tag{24}$$

and since optimality requires that $\dot{\lambda}_1$ and $\dot{\lambda}_2$ be of opposite sign [see eqn (6)] it follows that γ_1 and γ_2 must have the same sign. The same argument applies to the other principal values of $[\gamma_{ij}]$, and it has therefore been established that the definiteness of $[\gamma_{ij}]$ is also necessary for optimality.

We conclude by noting that whereas by eqn (23) the elements of the matrix $[\dot{\Lambda}_{ij}]$ are linearly dependent on the design change parameters $\dot{\alpha}_{kl}$, the same is not true of its principal values $\dot{\lambda}_i$. Nevertheless if $\dot{\alpha}_{kl}$ is replaced by $c\dot{\alpha}_{kl}$ then

$$\dot{\lambda}_i(c\dot{\alpha}_{kl}) = c\dot{\lambda}_i(\dot{\alpha}_{kl}). \tag{25}$$

In other words, within the $\dot{\alpha}_{kl}$ subspace of $\frac{1}{2}(n+2)(n-1)$ dimensions the n "surfaces" representing λ_i approach cones near the optimal design point $\lambda_i = \bar{\lambda}$, and eqn (4), in conjunction with the definiteness of $[\gamma_{ij}]$, insures that the cone corresponding to the smallest eigenvalue λ_1 lies "below" the plane $\lambda = \bar{\lambda}$.

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